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FUNCTIONS WITH SLOWLY-GROWING AREA AND HARMONIC MAJORANTS

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ABSTRACT

Let f be a function holomorphic in $U = \{|z| < 1\}$, and let $A(R, f)$ be the area of $f(U) \cap \{|w| < R\}$, not counting multiplicities. If $A(R, f) = O(R^{\gamma})$ as $R \to \infty$ for a γ, $0 \leqq \gamma < 2$, then the subharmonic function $\exp |f|^p$ has a harmonic majorant in U for each p, $0 < p < 2 - \gamma$. If $0 \leqq \gamma < 1$ further, then e^f is of Hardy class H^p for each p, $0 < p < \infty$.

1. Introduction

Let f be a function holomorphic in the disk $U = \{|z| < 1\}$, and let $D = f(U)$ be the image of U by f contained in the complex plane $\mathbf{C} = \{|w| < \infty\}$. Let $A(R, f)$ be the area of the intersection $D \cap U(R)$ of D and the disk $U(R) = \{|w| < R\}$, $R > 0$, so that $A(R, f) \leqq \pi R^2$. Then $A(R, f)$ is bounded if and only if D has the finite area. This condition is satisfied, for example, if

$$\iint_U |f'(z)|^2 dxdy < \infty,$$

the area of the Riemannian image of U by f covering over D.

A prototype of our present study is the result of L. J. Hansen and W. K. Hayman [3, theorem 1] (see [2]), that if

$$A(R, f) = o(R^2) \qquad \text{as } R \to \infty, \tag{1.1}$$

then f is of Hardy class H^p for all p, $0 < p < \infty$. Particularly, if $A(R, f)$ is bounded, then we have the same conclusion which, as Hansen claimed, extends the known fact, $f \in H^2$, resulting from [1, theorem 1]. Thus, one problem is: can we say any more about f with bounded $A(R, f)$?

For this purpose we consider a condition stronger than (1, 1), namely,

$$A(R, f) = O(R^{\gamma}) \qquad \text{as } R \to \infty, \tag{1.2}$$

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where γ is a constant with $0 \leqq \gamma < 2$. Evidently, f satisfies (1.2) with $\gamma = 0$ if $A(R, f)$ is bounded.

THEOREM 1. *Let f be holomorphic in U, and suppose that* (1.2) *holds for some γ, $0 \leqq \gamma < 2$. Then, for each p, $0 < p < 2 - \gamma$, the subharmonic function* $\exp|f|^p$ *admits a harmonic majorant u_p in U, that is, u_p is harmonic and* $\exp|f|^p \leqq u_p$ *in U.*

A comparison of the function $\exp x^p \equiv \exp(x^p)$ with x^q $(0 < q < \infty)$ for $x \geqq 0$, yields that $f \in H^q$ for all q, $0 < q < \infty$, provided that $\exp|f|^p$ has a harmonic majorant for some p, $0 < p < \infty$. In the case where $A(R, f)$ is bounded, $\exp|f|^p$ has a harmonic majorant for all p, $0 < p < 2$. We next consider the restrictive case where $0 \leqq \gamma < 1$ in

THEOREM 2. *Let f be holomorphic in U, and suppose that* (1.2) *holds for a γ, $0 \leqq \gamma < 1$. Then, e^f is of Hardy class H^p for each p, $0 < p < \infty$.*

Theorem 2 is sharp in the sense that the conclusion is false if (1.2) holds only for $\gamma \geqq 1$. An example is $f(z) = \log[(1+z)/(1-z)]$ for which (1.2) is true with $\gamma \geqq 1$, yet e^f is not of class H^1.

By uniformization, Theorems 1 and 2 have obvious extensions to Riemann surfaces W of hyperbolic type. One of the main interests on W concerns holomorphic f with finite Dirichlet integral

$$\iint_W |f'(z)|^2 \, dx dy < \infty.$$

Our version of Theorem 1, for example, says that $\exp|f|^p$ has a harmonic majorant on W for each p, $0 < p < 2$. For an earlier result that f is of class H^2 on W, see the paper of M. Parreau [4, p. 179].

In the proofs of Theorems 1 and 2, and in the other cases in the remainder of this paper, we always assume that f is an unbounded holomorphic function in U with $f(0) = 0$.

2. A test function

Let f be as in the last paragraph in the preceding section and let $a(t)$ be the angular Lebesgue measure of the longest subarc of the intersection $D \cap \{|w| = t\}$ $(D = f(U))$, so that the arc has the length $ta(t)$, and let the function $X(t)$ be zero if $\{|w| = t\}$ is contained in D, and be one otherwise $(0 < t < \infty)$. It is easy to observe that $a(t)$ is positive and lower-semicontinuous in $(0, \infty)$. A test function we shall consider is

$$F(R,f)=\int_1^R \frac{X(t)}{ta(t)}dt, \qquad R>1.$$

THEOREM 3. *Suppose that*

$$R^{-q}F(R,f)\to\infty \qquad \text{as } R\to\infty \tag{2.1}$$

for a q, $0<q<\infty$. *Then* $\exp|f|^p$ *has a harmonic majorant for each* p, $0<p<q$.

Postponing the proof of Theorem 3, we deduce Theorems 1 and 2 from Theorem 3. It suffices to show that (1.2) implies (2.1) for each q, $0<q<2-\gamma$. Let $m(R)$ be the Lebesgue measure of the linear set $L=\{t;\ 0<t<R$ and $a(t)=2\pi\}$. Let $b(t)$ be the angular measure of $D\cap\{|w|=t\}$, so that $a(t)\leqq b(t)$, $0<t<\infty$. Then

$$A(R,f)=\int_0^R tb(t)dt\geqq\int_0^R ta(t)dt\geqq\int_L ta(t)dt$$

$$=2\pi\int_L tdt\geqq 2\pi\int_0^{m(R)} tdt=\pi m(R)^2,$$

because t is monotone on the interval $(0,R)$. Therefore, (1.2) for γ, $0\leqq\gamma<2$, implies that $m(R)=O(R^{\gamma/2})$ as $R\to\infty$. Since $R-m(R)\leqq\int_0^R X(t)dt$, it follows that, for R sufficiently large,

$$[R-1-m(R)]^2\leqq\left[\int_1^R X(t)dt\right]^2$$

$$\leqq\left[\int_1^R ta(t)dt\right]\left[\int_1^R\frac{X(t)}{ta(t)}dt\right]\leqq A(R,f)F(R,f),$$

whence (2.1) is valid for q, $0<q<2-\gamma$.

For the proof of Theorem 2, we fix p, $1<p<2-\gamma$. Then, for each q, $0<q<\infty$, there exists a constant $c_q>0$ such that $qx\leqq x^p+c_q$ for $x\geqq 0$. Since

$$|e^f|^q\leqq e^{q|f|}\leqq(\exp c_q)\exp|f|^p,$$

it follows that $e^f\in H^q$.

For the proof of Theorem 3, we fix an arbitrary p, $0<p<q$. Our aim is to show that there exists a harmonic function h in D such that

$$\exp|w|^p\leqq h(w) \qquad \text{for all } w\in D. \tag{2.2}$$

If (2.2) is established, then $\exp|f(z)|^p$ has a harmonic majorant $h(f(z))$ in U.

To prove (2.2) we consider the estimate (3.3) of a harmonic measure in the next section. In Section 4, we construct h with the aid of another harmonic measure.

3. An estimate of a harmonic measure

Let $D(R)$ be the connected component of the intersection $D \cap U(R)$ such that $0 \in D(R)$, $R > 0$. Let u_R be the harmonic measure [6, p. 111] of $\{|w| = R\} \cap \partial D(R)$ with respect to the domain $D(R)$. Set

$$D^*(R) = U(R/4) \cap D(R) \qquad (R > 0). \tag{3.1}$$

Then the following estimate of the harmonic function u_R in $D^*(R)$ is known:

$$u_R(w) \leqq k \exp\left[-\pi \int_{2|w|}^{R/2} \frac{X(t)}{ta(t)} dt \right], \tag{3.2}$$

where w lies in $D^*(R)$, and k is a constant independent of w and $R > 0$. For the proof, see [5, theorem 2]. Those who cannot obtain Tsuji's paper [5] may consult [6, pp. 112–117]; an obvious version of the proof of a somewhat weaker result [6, corollary, p. 116] also proves (3.2).

Now, by (2.1), $\pi F(R/2, f) > R^q$ for all R larger than a constant $R_0 > 0$. It then follows from (3.2) that

$$u_R(w) \leqq c(w) \exp(-R^q) \tag{3.3}$$

for all w in $D^*(R)$ of (3.1) with $R > R_0$, where

$$c(w) = k \exp\left[-\pi \int_{2|w|}^{1} \frac{X(t)}{ta(t)} dt \right]$$

is a function of $w \in \mathbf{C}$.

We next show that the function c is bounded in each disk $U(R)$, $R > 0$. This is apparent for $R \leqq 1/2$ because $c(w) \leqq k$ for $2|w| \leqq 1$. In the case $R > 1/2$, we consider

$$\int_1^{2|w|} \frac{X(t)}{ta(t)} dt \leqq \int_1^{2R} \frac{X(t)}{ta(t)} dt$$

for $1/2 \leqq |w| < R$. The function $a(t)$ is positive and lower-semicontinuous in $(0, \infty)$, so that $1/a$ is bounded in $[1, 2R]$. Since $X(t)/[ta(t)] \leqq 1/a(t)$ in $[1, 2R]$, the integrand $X(t)/[ta(t)]$ is bounded in $[1, 2R]$, which shows that c is bounded in the annulus $\{1/2 \leqq |w| < R\}$, and hence in $U(R) = U(1/2) \cup \{1/2 \leqq |w| < R\}$.

4. The construction of h

To construct h in (2.2), we let v_R be the harmonic measure of $\{|w| \geqq R\} \cap \partial D$ with respect to the domain D $(R > 0)$; note that v_R is harmonic in D, while u_R is harmonic in $D(R) \subset D$. The maximum principle [6, theorem III.28, p. 77] then asserts that $v_R \leqq u_R$ in $D(R)$, so that (3.3) yields the estimate

$$v_R(w) \leqq c(w)\exp(-R^q) \tag{4.1}$$

for all w in $D^*(R)$ of (3.1) with $R > R_0$.

Set

$$h(w) = p\int_0^\infty v_R(w) R^{p-1} \exp R^p dR + 1 \tag{4.2}$$

for $w \in D$. To show that h is harmonic in D, we consider the Riemann integral

$$h_N(w) = p\int_0^N v(w, R) R^{p-1} \exp R^p dR + 1$$

for each $N = 1, 2, \cdots$ and each $w \in D$, where we set $v(w, R) \equiv v_R(w)$ for typographical reasons. Clearly, h_N is the limit

$$h_N(w) = \lim_{n\to\infty} S_n(w) \qquad (w \in D)$$

of the Riemann sum

$$S_n(w) = p\sum_{j=1}^{2^n} \frac{N}{2^n} v(w, R_j) R_j^{p-1} \exp R_j^p + 1, \qquad R_j = (2j-1)N/2^{n+1}.$$

The sequence $\{S_n\}$ of harmonic functions in D is uniformly bounded in D because $v(w, R_j) \geqq 1$, and

$$\lim_{n\to\infty} p\sum_{j=1}^{2^n} \frac{N}{2^n} R_j^{p-1} \exp R_j^p + 1 = \exp N^p.$$

Thus, $\{S_n\}$ is a normal family in D, so that h_N is the limit of a sequence of harmonic functions convergent locally and uniformly in D. Therefore h_N is harmonic in D. Since h_N is non-decreasing as $N \to \infty$, h is harmonic in D by Harnack's theorem if the convergence of $\{h_N\}$ at a point $w \in D$ is established. Fix $w \in D$, and find $R(w) > R_0$ such that $w \in D^*(R)$ for all $R > R(w)$. Then the integral in (4.2) is finite by (4.1), together with

$$R^{p-1}\exp(R^p - R^q) = o(R^{-2}) \qquad \text{as } R \to \infty.$$

Another property of h which will be used is that h is bounded in each domain $D(R')$, $R' > 0$. For the proof we choose $R'' > R_0$ such that $D(R') \subset D^*(R)$ for all $R > R''$. Since c is bounded by a constant c', say, in $U(R')$, it follows from (4.1) that $v_R(w) \leqq c' \exp(-R^q)$ for $w \in D(R')$ and $R > R''$. Thus,

$$h(w) \leqq p\int_0^{R''} R^{p-1} \exp R^p dR + pc' \int_{R''}^\infty R^{p-1} \exp(R^p - R^q) dR + 1 < \infty$$

for all $w \in D(R')$ because $p < q$.

As a final property of h, we assert that there exists a subset E of ∂D of capacity zero such that, for each pont $\alpha \in \partial D - E$,

$$\liminf h(w) \geqq \exp|\alpha|^p \tag{4.3}$$

as $w \to \alpha$ within D. Actually, v_R, as the solution of the Dirichlet problem [6, p. 4] in D with the characteristic function of $\{|w| \geqq R\} \cap \partial D$ as the boundary function, admits E such that, as $w \to \alpha \in \partial D - E$,

$$v_R(w) \to \begin{cases} 1 & \text{if } R \leqq |\alpha|, \\ 0 & \text{if } |\alpha| < R, \end{cases}$$

so that, by the Fatou lemma,

$$\liminf h(w) \geqq \liminf p \int_0^{|\alpha|} v_R(w) R^{p-1} \exp R^p dR + 1$$
$$\geqq p \int_0^{|\alpha|} R^{p-1} \exp R^p dR + 1 = \exp|\alpha|^p.$$

We are ready to prove (2.2). Fix $w \in D$, and then choose $R > R_0$ such that $w \in D^*(R)$. Let V_R be the solution of the Dirichlet problem in the bounded domain $D(R)$ with the continuous function $\exp|\alpha|^p$ $(\alpha \in \partial D(R))$ as the boundary value. Then, except for a set of capacity zero on $\partial D(R)$, the superior limit of the bounded harmonic function $V_R(\zeta) - (\exp R^p)u_R(\zeta) - h(\zeta)$ as $\zeta \to \alpha$ within $D(R)$, is non-positive. By the maximum principle again, and by the fact that $\exp|\zeta|^p \leqq V_R(\zeta)$ for all $\zeta \in D(R)$, we obtain

$$\exp|\zeta|^p \leqq (\exp R^p)u_R(\zeta) + h(\zeta) \qquad \text{for all } \zeta \in D(R).$$

Setting $\zeta = w$, and letting $R \to \infty$, one observes by (3.3) that (2.2) holds.

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